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On the Coefficient Bound Estimates and Fekete-Szegö Problem for a Certain Class of Analytic Functions

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Abstract: In the present study, a certain subclass of analytic and univalent functions in the open unit disk in the complex plane is introduced and examined. Then, for the class introduced, we study coefficient bound estimates and investigate the Fekete-Szegö problem. Furthermore, we discuss some intriguing special cases of the results found.

Analitik Fonksiyonların Belirli Bir Sınıfı İçin Katsayı Sınır Tahminleri ve Fekete-Szegö Problemi Üzerine

Anahtar Kelimeler:
Fekete-Szegö problemi,
univalent fonksiyon,
salagean operatörü

Özet: Sunulan çalışmada, kompleks düzlemin açık birim diskinde analitik ve univalent fonksiyonların belirli bir alt sınıfı tanıtılıyor ve inceleniyor. Sonrasında tanıtılan sınıf için katsayı sınır tahminlerini çalışıyor ve Fekete-Szegö problemini inceliyoruz. Ayrıca bulunan sonuçların bazı ilginç özel durumlarını tartışıyoruz.

1. INTRODUCTION

Let A denote the class of all complex valued functions f given by

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \; z \in \mathbb{C}, \quad (1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. By S, we define the class of all univalent functions in A. For $\alpha \in [0,1)$, some of the important and well-investigated subclasses of S include the classes $S^*(\alpha)$ and $C(\alpha)$,

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respectively, starlike and convex function classes of order α in U.

For the functions f and g which are analytic in U, f is said to be subordinate to g and denoted as $f(z) \prec g(z)$, if there exists an analytic function ω such that

$$\omega(0)=0,\ |\omega(z)|<1 \text{ and } f(z)=g(\omega(z)).$$

As is known that the coefficient upper bound problem is one of the important subjects of the theory of geometric functions. Firstly, by Lewin (Lewin, 1967) was introduced a subclass of bi-univalent functions and obtained the estimate $|a_2| \le 1.51$ for the function belonging to this class. Subsequently, Brannan and Clunie (Brannan and Clunie, 1980) developed the result of Lewin to $|a_2| \le \sqrt{2}$ for the bi-univalent function f. Later, Netanyahu (Netanyahu, 1969) showed that $|a_2| \le \frac{4}{3}$ for this class functions. Brannan and Taha (Brannan and Taha, 1986) were introduced a certain subclasses of bi-univalent function class Σ, namely bi-starlike function of order α denoted $S_\Sigma^*(\alpha)$ and bi-convex function of order α denoted $C_\Sigma(\alpha)$, respectively. For each of the function classes $S_\Sigma^*(\alpha)$ and $C_\Sigma(\alpha)$, non-sharp estimates on the first two coefficients were found by Brannan and Taha (Brannan and Taha, 1986). Many researchers have introduced and investigated several interesting subclasses of bi-univalent function class Σ and they have found non-sharp estimates on the first two coefficients (see Srivastava et al., 2010; Xu et al., 2012).

It is well known that the important tools in the theory of analytic functions is the functional $H_2(1) = a_3 - a_2^2$, which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $H_2(1) = a_3 - \mu a_2^2$, where μ is a complex or real number (see Fekete and Szegö, 1983). Estimating the upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegö problem in the theory of analytic functions. The Fekete-Szegö problem has been investigated by many mathematicians for several subclasses of analytic functions (see Mustafa, 2017; Mustafa and Gündüz, 2019; Zaprawa, 2014). Very soon, Mustafa and Mrugusundaramoorthy (Mustafa and Mrugusundaramoorthy, 2021) examine the Fekete-Szegö problem for the subclass of bi-univalent functions related to shell shaped region.

For the analytic function $f \in A$, Salagean (Salagean, 1983) introduced the following differential operator, which is called the Salagean operator

$$S^0 f(z) = f(z),\ S^1 f(z) = zSf(z) = zf'(z),$$

$$\begin{aligned} S^2 f(z) &= zS(Sf(z)) \\ &= zf''(z), \ldots, S^n f(z) \\ &= zS\left(S^{n-1} f(z)\right),\ n = 1, 2, \ldots . \end{aligned}$$

It follows from that

$$S^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,$$
$$z \in U,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Now, let we define the following subclass of analytic and univalent functions.

Definition 1.1. A function $f \in S$ is said to be in the class $C(n, \varphi)$ if the following condition is satisfied

$$1 + \frac{z\left(S^n f(z)\right)''}{\left(S^n f(z)\right)'} \prec \varphi(z),\ z \in U .$$

In this definition $\varphi(z) = z + \sqrt{1+z^2}$ and the branch of the square root is chosen to be principal one, that $\varphi(0) = 1$. It can be easily seen that the function $\varphi(z) = z + \sqrt{1+z^2}$ maps the unit disc U onto a shell shaped region on the right half plane and it is analytic and univalent in U. The range $\varphi(U)$ is symmetric respect to real axis and φ is a function with positive real part in U, with $\varphi(0) = \varphi'(0) = 1$. Moreover, it is a starlike domain with respect to point $\varphi(0) = 1$.

In the case $n = 0$, from the Definition 1.1 we have the subclass $C(0, \varphi) = C(\varphi)$.

Let, P be the set of the functions $p(z)$ analytic in U and satisfying $\operatorname{Re}(p(z)) > 0,\ z \in U$ and $p(0) = 1$, with power series

$$\begin{aligned} p(z) &= 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} p_n z^n,\ z \in U. \end{aligned}$$

In order to prove our main results in this paper, we shall need the following lemmas (see Duren, 1983; Grenander and Szegö, 1958).

Lemma 1.2. Let $p \in \mathrm{P}$, then $|p_n| \le 2$, $n = 1, 2, 3, \ldots$. These inequalities are sharp. In particular, equality holds for the function

$$p(z) = \frac{1+z}{1-z}$$

for all $n = 1, 2, 3, \ldots$.

Lemma 1.3. Let $p \in \mathrm{P}$, then $|p_n| \le 2$, $n = 1, 2, 3, \ldots$ and

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right) p_1 x$$
$$- 2\left(4 - p_1^2\right) p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z$$

for some x and z with $|x| < 1$ and $|z| < 1$.

Lemma 1.4. Let $p \in \mathrm{P}$, $B \in [0,1]$ and $B(2B-1) \le D \le B$.

Then, $\left|p_3 - 2Bp_1p_2 + Dp_1^3\right| \le 2$.

Remark 1.5. As can be seen from the serial expansion of the function φ given in Definition 1.1, this function belong to the class P.

In this paper, we give coefficient bound estimates and examine the Fekete-Szegö problem for the class $C(n,\varphi)$.

1. MAIN RESULTS

In this section, firstly we give the following theorem on the coefficient bound estimates for the class $C(n,\varphi)$.

Theorem 2.1. Let the function f given by (1) be in the class $C(n,\varphi)$. Then,

$$|a_2| \le \frac{1}{2^{n+1}}, \quad |a_3| \le \frac{1}{4\cdot 3^n} \text{ and } |a_4| \le \frac{5}{3\cdot 2^{2n+3}}.$$

Proof. Let $f \in C(n,\varphi)$. Then, according to Definition 1.1 there exists analytic function $\omega : U \to U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying the following condition

$$1 + \frac{z\left(S^n f(z)\right)''}{\left(S^n f(z)\right)'} = \varphi(\omega(z))$$
$$= \omega(z) + \sqrt{1 + \omega^2(z)}, \; z \in U. \tag{2}$$

Now, we define the function $p \in \mathrm{P}$ as follows

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)}$$
$$= 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} p_n z^n, \; z \in U.$$

From here, we write the following equality for the function ω

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1}$$
$$= \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^2}{4} \right) z^3 + \cdots \right],$$
$$z \in U. \tag{3}$$

Changing the expression of the function $\omega(z)$ in (2) with expressions in (3), we obtain

$$1+\frac{z\left(S^n f(z)\right)''}{\left(S^n f(z)\right)'} = 1+\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1 p_2}{4}\right)z^3+\cdots, \quad z\in U. \tag{4}$$

If the operations and simplifications on the left side of (4) are made and the coefficients of the terms of the same degree are equalized, are obtained the following equalities for a_2, a_3 and a_4

$$2^{n+1}a_2=\frac{p_1}{2},\ 2\cdot 3^{n+1}a_3-4^{n+1}a_2^2=\frac{p_2}{2}-\frac{p_1^2}{8},$$

$$3\cdot 4^{n+1}a_4-3\cdot 6^{n+1}a_2a_3+8^{n+1}a_2^3=\frac{p_3}{2}-\frac{p_1p_2}{4};$$

that is,

$$a_2=\frac{p_1}{2^{n+2}}, \tag{5}$$

$$a_3=\frac{1}{2}\left(\frac{4}{3}\right)^{n+1}a_2^2+\frac{1}{4\cdot 3^{n+1}}\left(p_2-\frac{p_1^2}{4}\right), \tag{6}$$

$$a_4=\left(\frac{3}{2}\right)^{n+1}a_2a_3-\frac{2^{n+1}}{3}a_2^3+\frac{1}{6\cdot 4^{n+1}}\left(p_3-\frac{p_1p_2}{2}\right). \tag{7}$$

By applying the Lemma 1.2 to equality (5), immediately obtained the first result of theorem.

Then, firstly using the Lemma 1.3 and applying triangle inequality and Lemma 1.2 to the equality (6), we get

$$|a_3|\le\frac{1}{8\cdot 3^{n+1}}\left[\frac{3}{2}t^2+\left(4-t^2\right)\xi\right],\ \xi\in(0,1)$$

with $\xi=|x|<1$. By maximizing the right-hand side of the last inequality with respect to the variable ξ, we obtain

$$|a_3|\le\frac{1}{8\cdot 3^{n+1}}\left(\frac{t^2}{2}+4\right),\ t\in[0,2].$$

Since, the function $\sigma=\frac{t^2}{2}+4,\ t\in[0,2]$ is an increasing function, from the last inequality obtained the second result of theorem.

Finally, let's we find an upper bound estimate for the coefficient a_4. From the equalities (5)-(7), we can write

$$a_4=\frac{1}{3\cdot 2^{2n+3}}\left[\frac{3}{4}p_1\left(p_2-\frac{p_1^2}{4}\right)+\left(p_3-\frac{p_1p_2}{2}+\frac{p_1^3}{6}\right)\right];$$

that is,

$$a_4=\frac{1}{3\cdot 2^{2n+3}}\left[\frac{3}{4}p_1\left(p_2-\frac{c}{2}p_1^2\right)+\left(p_3-2Bp_1p_2+Dp_1^3\right)\right],$$

with $c=\frac{1}{2}$, $B=\frac{1}{4}$ and $D=\frac{1}{6}$.

Using triangle equality to the last equality, we obtain

$$|a_4| \le \frac{1}{3\cdot 2^{2n+3}}\left[\frac{3}{4}|p_1|\left|p_2-\frac{c}{2}p_1^2\right|+\left|p_3-2Bp_1p_2+Dp_1^3\right|\right]. \tag{8}$$

Since $c=\frac{1}{2}\in[0,2]$, $B=\frac{1}{4}\in[0,1]$, $D=\frac{1}{6}$ and $B(2B-1)\le D\le B$, then according to Lemma 1.2 and Lemma 1.4, we write the following inequalities

$$\left|p_2-\frac{c}{2}p_1^2\right|\le 2 \text{ and } \left|p_3-2Bp_1p_2+Dp_1^3\right|\le 2,$$

respectively. Considering these inequalities, from the inequality (8) obtained desired estimate for the upper bound of $|a_4|$. With this, the proof of Theorem 2.1 is completed.

In the case $n=0$, from the Theorem 2.1 obtained the following result.

Corollary 2.2. Let $f\in C(\varphi)$, then

$$|a_2|\le\frac{1}{2},\ |a_3|\le\frac{1}{4} \text{ and and } |a_4|\le\frac{5}{24}.$$

Now, we give the following theorem on the Fekete-Szegö problem for the class $C(\varphi)$.

Theorem 2.3. Let the function f given by (1) be in the class $C(n,\varphi)$ and $\mu\in\mathbb{C}$. Then,

$$|a_3-\mu a_2^2|\le \frac{1}{4\cdot 3^{n+1}}\begin{cases} 2 & \text{if } \left|\left(\frac{4}{3}\right)^{n+1}-2\mu\right|\le \frac{1}{2}\left(\frac{4}{3}\right)^{n+1}, \\ 2\cdot\left(\frac{3}{4}\right)^{n+1}\left|\left(\frac{4}{3}\right)^{n+1}-2\mu\right|+1 & \text{if } \left|\left(\frac{4}{3}\right)^{n+1}-2\mu\right|> \frac{1}{2}\left(\frac{4}{3}\right)^{n+1}. \end{cases}$$

Proof. Let $f\in C(n,\varphi)$ and $\mu\in\mathbb{C}$. Then, from the expressions for the coefficients a_2 and a_3 in the equalities (5) and (6), we can write

$$a_3-\mu a_2^2=\frac{1}{2}\left\{\begin{array}{l}\left[\left(\frac{4}{3}\right)^{n+1}-2\mu\right]a_2^2+ \\ \frac{1}{2\cdot 3^{n+1}}\left(p_2-\frac{p_1^2}{4}\right)\end{array}\right\}.$$

Considering (5) and using Lemma 1.3, we write the above expression as follows

$$a_3-\mu a_2^2= \frac{1}{2\cdot 4^{n+2}}\left\{\begin{array}{l}\left[\left(\frac{4}{3}\right)^{n+1}-2\mu\right]p_1^2+ \\ \frac{1}{2}\left(\frac{4}{3}\right)^{n+1}\left[p_1^2+2\left(4-p_1^2\right)x\right]\end{array}\right\}.$$

for some x with $|x|<1$. From this, using triangle inequality we obtain

$$\left|a_3-\mu a_2^2\right| \le \frac{1}{2\cdot 4^{n+2}}\left\{\begin{array}{l}\left|\left(\frac{4}{3}\right)^{n+1}-2\mu\right|t^2+\\ \frac{1}{2}\left(\frac{4}{3}\right)^{n+1}\left[t^2+2\left(4-t^2\right)\xi\right]\end{array}\right\}, \xi\in(0,1)$$

with $\xi=|x|$. If we maximize the right-hand side of this inequality with respect to the parameter ξ, we get

$$\left|a_3-\mu a_2^2\right| \le \frac{1}{2\cdot 4^{n+2}}\left\{\left[\left|\left(\frac{4}{3}\right)^{n+1}-2\mu\right|-\frac{1}{2}\left(\frac{4}{3}\right)^{n+1}\right]t^2+4\cdot\left(\frac{4}{3}\right)^{n+1}\right\},$$
$$t\in[0,2]$$

.

Then, by maximizing the right-hand side of the last inequality with respect to the variable t, we arrive at the result of the theorem.

Thus, the proof of the Theorem 2.3 is completed.

In the case $\mu=0$ and $n=0$, from the Theorem 2.3 obtained the following results, respectively.

Corollary 2.4. Let $f\in C(n,\varphi)$, then

$$|a_3|\le\frac{1}{4\cdot 3^n}.$$

Corollary 2.5. Let $f\in C(\varphi)$, then

$$|a_3|\le\frac{1}{4}.$$

Remark 2.6. Result obtained in the Corollary 2.4 confirm the second inequality obtained in Theorem 2.1.

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