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Some Important Properties of Almost Kenmotsu $(\kappa,\mu,\nu)-$Space on the Concircular Curvature Tensor

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Abstract

In this article, pseudoparallel submanifolds for almost Kenmotsu $(\kappa,\mu,\nu)-$space are investigated. The almost Kenmotsu $(\kappa,\mu,\nu)-$space is considered on the concircular curvature tensor. Submanifolds of these manifolds with properties such as concircular pseudoparallel, concircular $2-$pseudoparallel, concircular Ricci generalized pseudoparallel, and concircular $2-$Ricci generalized pseudoparallel has been characterized. Necessary and sufficient conditions are given for the invariant submanifolds of almost Kenmotsu $(\kappa,\mu,\nu)-$space to be total geodesic according to the behavior of the κ,μ,ν functions.

1. Introduction

Let $\left(M^{2n+1},\phi,\xi,\eta,g\right)$ be a $(2n+1)-$dimensional contact metric manifold. We know that here R is the curvature tensor, ξ is the characteristic vector field and the condition $R(\rho_1,\rho_2)\xi=0$ is satisfied, for any vector field $\rho_1,\rho_2\in M^{2n+1}$. The contact metric manifold that satisfies this condition also satisfies the condition

$$R(\rho_1,\rho_2)\xi=\eta(\rho_2)(\kappa I+\mu h)\rho_1-\eta(\rho_1)(\kappa I+\mu h)\rho_2, \tag{1.1}$$

and this condition is called (κ,μ) nullity condition, where κ,μ are constants and h is the self adjoint $(1,1)-$tensor field. E. Boeckx in [1] and D. E. Blair et al. in [2], (κ,μ) nullity conditions on contact metric manifolds are considered when κ and μ are constant. E. Boeckx proved that non-Sasakian contact metric manifold is completely determined locally by its dimension for the constant values of κ and μ. If vector field ξ relate to the $(\kappa,\mu)-$nullity distribution, then (1.1) is provided and the manifold $\left(M^{2n+1},\phi,\xi,\eta,g\right)$ is described $(\kappa,\mu)-$contact metric manifold.
In particular, if κ and μ are not constant smooth functions on M^{2n+1}, then the manifold $\left(M^{2n+1},\phi,\xi,\eta,g\right)$ is described generalized $(\kappa,\mu)-$contact metric manifold [2].
T. Koufogiorgos et al. introduced $(\kappa,\mu,\nu)-$contact metric manifold in [3]. Riemann curvature tensor of $(\kappa,\mu,\nu)-$contact metric manifoldis in the form

$$\tilde{R}(\rho_1,\rho_2)\xi=\kappa[\eta(\rho_2)\rho_1-\eta(\rho_1)\rho_2]+\mu[\eta(\rho_2)h\rho_1-\eta(\rho_1)h\rho_2]+\nu[\eta(\rho_2)\phi h\rho_1-\eta(\rho_1)\phi h\rho_2], \tag{1.2}$$

for all $\rho_1,\rho_2\in\Gamma(TM)$, where κ,μ,ν are smooth functions on M^{2n+1}.
If $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$, then this manifold is an almost Kenmotsu manifold, where $\Phi(\rho_1,\rho_2)=g(\rho_1,\phi\rho_2)$ is the fundamental $2-$form of M^{2n+1}. If an almost Kenmotsu manifold provide a $(\kappa,\mu,\nu)-$nullity distribution, it is described an almost Kenmotsu $(\kappa,\mu,\nu)-$space [4].

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Later on, manifolds that do not have a contact metric structure but satisfy condition (1.2) have been studied. The almost cosymplectic $(\kappa,\mu,\nu)-$space is defined by P. Dacko and Z. Olszak in [5]. M. Atçeken obtained very important properties of almost Kenmotsu $(\kappa,\mu,\nu)-$space in [6]. Pseudoparallel submanifolds of many different structures have been investigated in [7–18].
The concept of submanifold for a manifold is quite interesting. For example, it plays a very important role in fields such as applied mathematics, analysis and physics, contributing to the illumination of these fields.
In this article, pseudoparallel submanifolds for almost Kenmotsu $(\kappa,\mu,\nu)-$space are investigated. The almost Kenmotsu $(\kappa,\mu,\nu)-$space is considered on the concircular curvature tensor. Submanifolds of these manifolds with properties such as concircular pseudoparallel, concircular $2-$pseudoparallel, concircular Ricci generalized pseudoparallel, and concircular $2-$Ricci generalized pseudoparallel has been characterized. Necessary and sufficient conditions are given for the invariant submanifolds of almost Kenmotsu $(\kappa,\mu,\nu)-$space to be total geodesic according to the behavior of the κ,μ,ν functions.

2. Preliminary

Let $\tilde{N}$ be $(2n+1)-$dimensional contact metric manifold. This manifold admits an almost contact metric structure (ϕ,ξ,η,g) such that

$$\phi^2\rho_1=-\rho_1+\eta(\rho_1)\xi,\quad \eta(\rho_1)=g(\rho_1,\xi),\quad \eta(\xi)=1,\eta\circ\phi=0, \tag{2.1}$$

$$g(\phi\rho_1,\phi\rho_2)=g(\rho_1,\rho_2)-\eta(\rho_1)\eta(\rho_2), \tag{2.2}$$

for all vector fields $\rho_1,\rho_2\in\Gamma\left(T\tilde{N}\right)$, where $\Gamma\left(T\tilde{N}\right)$ denotes the set of differentiable vector fields on $\tilde{N}$ [3]. $\tilde{N}$ together with the (ϕ,ξ,η,g) is called a contact metric manifold.
The Riemannian curvature tensor $\tilde{R}$ of $\tilde{N}$ is given

$$\tilde{R}(\rho_1,\rho_2)=\tilde{\nabla}_{\rho_1}\tilde{\nabla}_{\rho_2}-\tilde{\nabla}_{\rho_2}\tilde{\nabla}_{\rho_1}-\tilde{\nabla}_{[\rho_1,\rho_2]},$$

for all $\rho_1,\rho_2\in\Gamma\left(T\tilde{N}\right)$, where $\tilde{\nabla}$ is the Levi-Civita connection of g.
Let h be tensor field $(1,1)-$type and l_ξ be the Lie-derivative in the direction of ξ. Thus, we can write

$$2h\rho_1=\left(l_\xi\phi\right)\rho_1,$$

for all $\rho_1\in\Gamma\left(T\tilde{N}\right)$. On the other hand h is self-adjoint and satisfies

$$\phi h+h\phi=0, trh=tr\phi h=0, h\xi=0. \tag{2.3}$$

In addition, contact metric manifolds provide the formula given by

$$\tilde{\nabla}_{\rho_1}\xi=\phi\rho_1-\phi h\rho_1,\tilde{\nabla}_\xi\phi=0. \tag{2.4}$$

The $(\kappa,\mu)-$nullity distribution of a contact metric manifold $\tilde{N}$ for the pair $(\kappa,\mu)\in\mathbb{R}^2$ is distribution

$$\tilde{R}(\rho_1,\rho_2)\rho_3=\kappa[g(\rho_2,\rho_3)\rho_1-g(\rho_1,\rho_3)\rho_2]+\mu[g(\rho_2,\rho_3)h\rho_1-g(\rho_1,\rho_3)h\rho_2],$$

for all $\rho_1,\rho_2\in\Gamma\left(T\tilde{N}\right)$.
Now let's give some equations below which are important for almost Kenmotsu $(\kappa,\mu,\nu)-$space. Let $\tilde{N}^{2n+1}(\phi,\eta,\xi,g)$ be $(2n+1)-$dimensional almost Kenmotsu $(\kappa,\mu,\nu)-$space. Then the following relations are provided.

$$h^2=(\kappa+1)\phi^2,\kappa\leq-1, \tag{2.5}$$

$$\xi(\kappa)=2(\kappa+1)(\nu-2), \tag{2.6}$$

$$\left(\tilde{\nabla}_{\rho_1}\phi\right)\rho_2=g(\phi\rho_1+h\rho_1,\rho_2)\xi-\eta(\rho_2)(\phi\rho_1+h\rho_1), \tag{2.7}$$

$$\tilde{\nabla}_{\rho_1}\xi=-\phi^2\rho_1-\phi h\rho_1, \tag{2.8}$$

$$S(\rho_1,\xi)=2n\kappa\eta(\rho_1), \tag{2.9}$$

$$\tilde{R}(\xi,\rho_1)\rho_2=\kappa[g(\rho_1,\rho_2)\xi-\eta(\rho_2)\rho_1]+\mu[g(h\rho_1,\rho_2)\xi-\eta(\rho_2)h\rho_1]+\nu[g(\phi h\rho_1,\rho_2)\xi-\eta(\rho_2)\phi h\rho_1]. \tag{2.10}$$

Let N be the immersed submanifold of an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Let the tangent and normal subspaces of N in $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$ be $\Gamma(TN)$ and $\Gamma\left(T^{\perp}N\right)$, respectively. Gauss and Weingarten formulas for $\Gamma(TM)$ and $\Gamma\left(T^{\perp}M\right)$ are

$$\tilde{\triangledown}_{\rho_1}\rho_2=\triangledown_{\rho_1}\rho_2+\sigma(\rho_1,\rho_2), \tag{2.11}$$

$$\tilde{\triangledown}_{\rho_1}\rho_5=-A_{\rho_5}\rho_1+\triangledown^{\perp}_{\rho_1}\rho_5, \tag{2.12}$$

respectively, for all $\rho_1,\rho_2\in\Gamma\left(T\tilde{M}\right)$ and $\rho_5\in\Gamma\left(T^{\perp}\tilde{M}\right)$, where $\triangledown$ and $\triangledown^{\perp}$ are the connections on N and $\Gamma\left(T^{\perp}N\right)$, respectively, σ and A are the second fundamental form and the shape operator of N. There is a relation

$$g\left(A_{\rho_5}\rho_1,\rho_2\right)=g\left(\sigma(\rho_1,\rho_2),\rho_5\right) \tag{2.13}$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form σ is defined as

$$\left(\tilde{\triangledown}_{\rho_1}\sigma\right)(\rho_2,\rho_3)=\triangledown^{\perp}_{\rho_1}\sigma(\rho_2,\rho_3)-\sigma\left(\triangledown_{\rho_1}\rho_2,\rho_3\right)-\sigma\left(\rho_2,\triangledown_{\rho_1}\rho_3\right), \tag{2.14}$$

for all $\rho_1,\rho_2,\rho_3\in\Gamma(TN)$. Specifically, if $\tilde{\triangledown}\sigma=0$, N is said to be its second fundamental form is parallel.
Let R be the Riemann curvature tensor of N. In this case, the Gauss equation can be expressed as

$$\tilde{R}(\rho_1,\rho_2)\rho_3=R(\rho_1,\rho_2)\rho_3+A_{\sigma(\rho_1,\rho_3)}\rho_2-A_{\sigma(\rho_2,\rho_3)}\rho_1+\left(\tilde{\triangledown}_{\rho_1}\sigma\right)(\rho_2,\rho_3)-\left(\tilde{\triangledown}_{\rho_2}\sigma\right)(\rho_1,\rho_3), \tag{2.15}$$

for all $\rho_1,\rho_2,\rho_3\in\Gamma(TN)$.
$\tilde{R}\cdot\sigma$ is given by

$$\left(\tilde{R}(\rho_1,\rho_2)\cdot\sigma\right)(\rho_4,\rho_5)=R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma\left(R(\rho_1,\rho_2)\rho_4,\rho_5\right)-\sigma\left(\rho_4,R(\rho_1,\rho_2)\rho_5\right), \tag{2.16}$$

where the Riemannian curvature tensor of normal bundle $\Gamma\left(T^{\perp}N\right)$ is given

$$R^{\perp}(\rho_1,\rho_2)=\left[\nabla^{\perp}_{\rho_1},\nabla^{\perp}_{\rho_2}\right]-\nabla^{\perp}_{[\rho_1,\rho_2]}.$$

On the other hand, the concircular curvature tensor for Riemannian manifold $\left(N^{2n+1},g\right)$ is given by

$$C(\rho_1,\rho_2)\rho_3=\tilde{R}(\rho_1,\rho_2)\rho_3-\frac{r}{2n(2n+1)}\left[g(\rho_2,\rho_3)\rho_1-g(\rho_1,\rho_3)\rho_2\right], \tag{2.17}$$

where r denotes the scalar curvature of N.
Similarly, the tensor $C\cdot\sigma$ is defined by

$$\left(C(\rho_1,\rho_2)\cdot\sigma\right)(\rho_4,\rho_5)=R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma\left(C(\rho_1,\rho_2)\rho_4,\rho_5\right)-\sigma\left(\rho_4,C(\rho_1,\rho_2)\rho_5\right), \tag{2.18}$$

for all $\rho_1,\rho_2,\rho_4,\rho_5\in\Gamma(TN)$.
Let N be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,...,X_k;\rho_1,\rho_2)=-T\left((\rho_1\wedge_A\rho_2)X_1,...,X_k\right)-...-T\left(X_1,...,X_{k-1},(\rho_1\wedge_A\rho_2)X_k\right), \tag{2.19}$$

where

$$(\rho_1\wedge_A\rho_2)\rho_3=A(\rho_2,\rho_3)\rho_1-A(\rho_1,\rho_3)\rho_2, \tag{2.20}$$

$k\geq 1,X_1,X_2,...,X_k,\rho_1,\rho_2\in\Gamma(TN)$.

Definition 2.1 ([8]). *A submanifold N of a Riemannian manifold $\left(\tilde{N},g\right)$ is said to be concircular pseudoparallel, concircular $2-$pseudoparallel, concircular Ricci-generalized pseudoparallel and concircular $2-$Ricci generalized pseudoparallel if*

$C\cdot\sigma$ *and* $Q(g,\sigma)$

$C\cdot\tilde{\nabla}\sigma$ *and* $Q\left(g,\tilde{\nabla}\sigma\right)$

$C\cdot\sigma$ *and* $Q(S,\sigma)$

$C\cdot\tilde{\nabla}\sigma$ *and* $Q\left(S,\tilde{\nabla}\sigma\right)$

are linearly dependent, respectively.

3. Invariant Pseudoparalel Submanifolds of an Almost Kenmotsu $(\kappa,\mu,\nu)-$Space

Let N be the immersed submanifold of an $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If $\phi\left(T_{\rho_1}N\right)\subset T_{\rho_1}N$ in every ρ_1 point, the manifold N is called invariant submanifold. We note that all of properties of an invariant submanifold inherit the ambient manifold. From this section of the article, we will assume that the manifold N is the invariant submanifold of the an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. So, it is clear that the following proposition.

Proposition 3.1. *Let N be an invariant submanifold of an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$ such that ξ is tangent to N. Then the following equalities hold on N.*

$$R(\rho_1,\rho_2)\xi=\kappa[\eta(\rho_2)\rho_1-\eta(\rho_1)\rho_2]+\mu[\eta(\rho_2)h\rho_1-\eta(\rho_1)h\rho_2]+\nu[\eta(\rho_2)\phi h\rho_1-\eta(\rho_1)\phi h\rho_2], \tag{3.1}$$

$$R(\xi,\rho_1)\rho_2=\kappa[g(\rho_1,\rho_2)\xi-\eta(\rho_2)\rho_1]+\mu[g(h\rho_1,\rho_2)\xi-\eta(\rho_2)h\rho_1]+\nu[g(\phi h\rho_1,\rho_2)\xi-\eta(\rho_2)\phi h\rho_1], \tag{3.2}$$

$$\left(\nabla_{\rho_1}\phi\right)\rho_2=g(\phi\rho_1+h\rho_1,\rho_2)\xi-\eta(\rho_4)(\phi\rho_1+h\rho_1), \tag{3.3}$$

$$\nabla_{\rho_1}\xi=-\phi^2\rho_1-\phi h\rho_1, \tag{3.4}$$

$$C(\rho_1,\rho_2)\xi=\left[\kappa-\frac{r}{2n(2n+1)}\right][\eta(\rho_2)\rho_1-\eta(\rho_1)\rho_2]+\mu[\eta(\rho_2)h\rho_1-\eta(\rho_1)h\rho_2]+\nu[\eta(\rho_2)\phi h\rho_1-\eta(\rho_1)\phi h\rho_2], \tag{3.5}$$

$$\begin{aligned}C(\xi,\rho_1)\rho_2=&\left[\kappa-\frac{r}{2n(2n+1)}\right][g(\rho_1,\rho_2)\xi-\eta(\rho_2)\rho_1]+\mu[g(h\rho_1,\rho_2)\xi-\eta(\rho_2)h\rho_1]\\&+\nu[g(\phi h\rho_1,\rho_2)\xi-\eta(\rho_2)\phi h\rho_1],\end{aligned} \tag{3.6}$$

$$C(\xi,\rho_1)\xi=\left[\kappa-\frac{r}{2n(2n+1)}\right][\eta(\rho_1)\xi-\rho_1] \tag{3.7}$$

$$\sigma(\rho_1,\xi)=0,\quad \sigma(\phi\rho_1,\rho_2)=\sigma(\rho_1,\phi\rho_2)=\phi\sigma(\rho_1,\rho_2), \tag{3.8}$$

for all $\rho_1,\rho_2\in\Gamma(TN)$, where ∇,σ and R denote the induced Levi-Civita connection on N, the shape operator and Riemannian curvature tensor of N, respectively.

Lemma 3.2 ([6]). *Let N be the invariant submanifold of an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Then the second fundamental form σ of N is parallel if and only if N is the total geodesic submanifold provided $\kappa\neq 0$.*

Let us now consider the invariant submanifolds of the almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$ on the concircular curvature tensor.
Equivalent to the definition of concircular pseudoparallel given above, it can be said that there is a function $\digamma_1$ on the set $M_1=\{x\in N|\,\sigma(x)\neq g(x)\}$ such that

$$C\cdot\sigma=\digamma_1 Q(g,\sigma).$$

If $\digamma_1=0$ specifically, N is called a concircular semiparallel submanifold.

Theorem 3.3. *Let N be the invariant submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If N is concircular pseudoparallel submanifold, then N is either a total geodesic submanifold or*

$$\digamma_1=\left(\kappa-\frac{r}{2n(2n+1)}\right)\mp\sqrt{(\kappa+1)(\nu^2-\mu^2)},\mu\cdot\nu(\kappa+1)=0.$$

Proof. Let's assume that N is a concircular pseudoparallel submanifold. So, we can write

$$(C(\rho_1,\rho_2)\cdot\sigma)(\rho_4,\rho_5)=\digamma_1 Q(g,\sigma)(\rho_4,\rho_5;\rho_1,\rho_2), \tag{3.9}$$

for all $\rho_1,\rho_2,\rho_4,\rho_5\in\Gamma(TN)$. From (2.18), it is clear that

$$R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma(C(\rho_1,\rho_2)\rho_4,\rho_5)-\sigma(\rho_4,C(\rho_1,\rho_2)\rho_5)=-\digamma_1\left\{\sigma((\rho_1\wedge_g\rho_2)\rho_4,\rho_5)+\sigma(\rho_4,(\rho_1\wedge_g\rho_2)\rho_5)\right\}.$$

Easily from here, we can write

$$R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma(C(\rho_1,\rho_2)\rho_4,\rho_5)-\sigma(\rho_4,C(\rho_1,\rho_2)\rho_5)=-\digamma_1\{g(\rho_2,\rho_4)\sigma(\rho_1,\rho_5)-g(\rho_1,\rho_4)\sigma(\rho_2,\rho_5)$$
$$+g(\rho_2,\rho_5)\sigma(\rho_4,\rho_1)-g(\rho_1,\rho_5)\sigma(\rho_4,\rho_2)\}. \tag{3.10}$$

If we choose $\rho_1=\rho_4=\xi$ in (3.10) and make use of (3.7), we get

$$\sigma(C(\xi,\rho_2)\xi,\rho_5)=-\digamma_1\sigma(\rho_2,\rho_5). \tag{3.11}$$

If we use (3.7) out of (3.11), we obtain

$$\left[\digamma_1-\left(\kappa-\frac{r}{2n(2n+1)}\right)\right]\sigma(\rho_2,\rho_5)=\mu\sigma(h\rho_2,\rho_5)+\nu\phi\sigma(h\rho_2,\rho_5). \tag{3.12}$$

Substituting $h\rho_2$ for ρ_2 in (3.12) by view of (2.5) and (3.8), we have

$$\left[\digamma_1-\left(\kappa-\frac{r}{2n(2n+1)}\right)\right]\sigma(h\rho_2,\rho_5)=-(\kappa+1)[\mu\sigma(\rho_2,\rho_5)+\nu\phi\sigma(\rho_2,\rho_5)]. \tag{3.13}$$

From (3.12) and (3.13), one can easily see that

$$\left\{(\kappa+1)(\mu^2-\nu^2)+\left[\digamma_1-\left(\kappa-\frac{r}{2n(2n+1)}\right)\right]^2\right\}\sigma(\rho_2,\rho_5)+2(\kappa+1)\mu\nu\phi\sigma(\rho_2,\rho_5)=0. \tag{3.14}$$

This tell us that N is either totally geoesic submanifold or

$$(\kappa+1)(\mu^2-\nu^2)+\left[\digamma_1-\left(\kappa-\frac{r}{2n(2n+1)}\right)\right]^2=(\kappa+1)\mu\nu=0.$$

This completes the proof. □

Corollary 3.4. *Let N be an invariant pseudoparallel submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Then N is concircular semiparallel if and only if N is totally geodesic provided*

$$(\kappa+1)(\mu^2-\nu^2)+\left(\kappa-\frac{r}{2n(2n+1)}\right)^2\neq 0 \text{ or } (\kappa+1)\mu\nu\neq 0.$$

Equivalent to the definition of concircular Ricci generalized pseudoparallel given above, it can be said that there is a function $\digamma_2$ on the set
$M_2=\{x\in N|S(x)\neq\sigma(x)\}$ such that

$$C\cdot\sigma=\digamma_2 Q(S,\sigma).$$

If $\digamma_2=0$ specifically, N is called a concircular Ricci generalized semiparallel submanifold.

Theorem 3.5. *Let N be the invariant submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If N is concircular Ricci generalized pseudoparallel submanifold, then N is either a total geodesic submanifold or*

$$\digamma_2=\frac{2n(2n+1)\kappa-r}{4n^2\kappa(2n+1)}\mp\frac{1}{2n\kappa}\sqrt{(\kappa+1)(\nu^2-\mu^2)},\mu\cdot\nu(\kappa+1)=0.$$

Proof. Let's assume that N is a concircular Ricci generalized pseudoparallel submanifold. So, we can write

$$(C(\rho_1,\rho_2)\cdot\sigma)(\rho_4,\rho_5)=\digamma_2 Q(S,\sigma)(\rho_4,\rho_5;\rho_1,\rho_2), \tag{3.15}$$

for all $\rho_1,\rho_2,\rho_4,\rho_5\in\Gamma(TN)$. From (2.18), it is clear that

$$R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma(C(\rho_1,\rho_2)\rho_4,\rho_5)-\sigma(\rho_4,C(\rho_1,\rho_2)\rho_5)=-\digamma_2\{\sigma((\rho_1\wedge_S\rho_2)\rho_4,\rho_5)+\sigma(\rho_4,(\rho_1\wedge_S\rho_2)\rho_5)\}.$$

Easily from here, we can write

$$R^{\perp}(\rho_1,\rho_2)\sigma(\rho_4,\rho_5)-\sigma(C(\rho_1,\rho_2)\rho_4,\rho_5)-\sigma(\rho_4,C(\rho_1,\rho_2)\rho_5)=-\digamma_2\{S(\rho_2,\rho_4)\sigma(\rho_1,\rho_5)-S(\rho_1,\rho_4)\sigma(\rho_2,\rho_5)$$
$$+S(\rho_2,\rho_5)\sigma(\rho_4,\rho_1)-S(\rho_1,\rho_5)\sigma(\rho_4,\rho_2)\}. \tag{3.16}$$

If we choose $\rho_1=\rho_5=\xi$ in (3.16) and make use of (3.7), we get

$$\sigma(\rho_4, C(\xi,\rho_2)\xi) = -\digamma_2 S(\xi,\xi)\sigma(\rho_4,\rho_2). \tag{3.17}$$

If we use (2.9) and (3.7) in (3.17), we obtain

$$\left[\left(\kappa-\frac{r}{2n(2n+1)}\right)-2n\kappa\digamma_2\right]\sigma(\rho_4,\rho_2)=\mu\sigma(\rho_4,h\rho_2)+\nu\phi\sigma(\rho_4,h\rho_2). \tag{3.18}$$

Substituting $h\rho_2$ for ρ_2 in (3.18) by view of (2.5) and (3.8), we have

$$\left[\left(\kappa-\frac{r}{2n(2n+1)}\right)-2n\kappa\digamma_2\right]\sigma(h\rho_2,\rho_4)=-(\kappa+1)\left[\mu\sigma(\rho_2,\rho_4)+\nu\phi\sigma(\rho_2,\rho_4)\right]. \tag{3.19}$$

From (3.18) and (3.19), one can easily see that

$$\left\{\left[\left(\kappa-\frac{r}{2n(2n+1)}\right)-2n\kappa\digamma_2\right]^2+(\kappa+1)\left(\mu^2-\nu^2\right)\right\}\sigma(\rho_4,\rho_2)+2(\kappa+1)\mu\nu\phi\sigma(\rho_4,\rho_2)=0. \tag{3.20}$$

This tell us that N is either totally geoesic submanifold or

$$\left[\left(\kappa-\frac{r}{2n(2n+1)}\right)-2n\kappa\digamma_2\right]^2+(\kappa+1)\left(\mu^2-\nu^2\right)=(\kappa+1)\mu\nu=0.$$

This completes the proof. □

Corollary 3.6. *Let N be an invariant pseudoparallel submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Then N is concircular Ricci generalized semiparallel if and only if N is totally geodesic provided*

$$(\kappa+1)\left(\mu^2-\nu^2\right)+\left(\kappa-\frac{r}{2n(2n+1)}\right)^2\neq 0 \text{ or } (\kappa+1)\mu\nu\neq 0.$$

Equivalent to the definition of concircular $2-$pseudoparallel given above, it can be said that there is a function $\digamma_3$ on the set $M_3=\left\{x\in N|\, g(x)\neq\tilde{\nabla}\sigma(x)\right\}$ such that

$$C\cdot\tilde{\nabla}\sigma=\digamma_3 Q\left(g,\tilde{\nabla}\sigma\right).$$

If $\digamma_3=0$ specifically, N is called a concircular $2-$semiparallel submanifold.

Theorem 3.7. *Let N be the invariant submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If N is concircular $2-$pseudoparallel submanifold, then N is either a total geodesic submanifold or*

$$\digamma_3=\left[\kappa-\frac{r}{2n(2n+1)}\right]\mp\sqrt{(\kappa+1)(\nu^2-\mu^2)},\mu\cdot\nu(\kappa+1)=0.$$

Proof. Let's assume that $\tilde{M}$ is a concircular $2-$pseudoparallel submanifold. So, we can write

$$\left(C(\rho_1,\rho_2)\cdot\tilde{\triangledown}\sigma\right)(\rho_4,\rho_5,\rho_3)=\digamma_3 Q\left(S,\tilde{\triangledown}\sigma\right)(\rho_4,\rho_5,\rho_3;\rho_1,\rho_2), \tag{3.21}$$

for all $\rho_1,\rho_2,\rho_4,\rho_5,\rho_3\in\Gamma(TM)$. If we choose $\rho_1=\rho_5=\xi$ in (3.21), we can write

$$\begin{aligned}&R^{\perp}(\xi,\rho_2)\left(\tilde{\triangledown}_{\rho_4}\sigma\right)(\xi,\rho_3)-\left(\tilde{\triangledown}_{C(\xi,\rho_2)\rho_4}\sigma\right)(\xi,\rho_3)-\left(\tilde{\triangledown}_{\rho_4}\sigma\right)(C(\xi,\rho_2)\xi,\rho_3)-\left(\tilde{\triangledown}_{\rho_4}\sigma\right)(\xi,C(\xi,\rho_2)\rho_3)\\&=-\digamma_3\left\{\left(\tilde{\triangledown}_{(\xi\wedge_g\rho_2)\rho_4}\sigma\right)(\xi,\rho_3)+\left(\tilde{\triangledown}_{\rho_4}\sigma\right)((\xi\wedge_g\rho_2)\xi,\rho_3)+\left(\tilde{\triangledown}_{\rho_4}\sigma\right)(\xi,(\xi\wedge_g\rho_2)\rho_3)\right\}.\end{aligned} \tag{3.22}$$

Let's calculate all the expressions in (3.22). In view of (2.14), (2.19), (3.4), and (3.8), we can derive

$$\begin{aligned}R^{\perp}(\xi,\rho_2)\left(\tilde{\triangledown}_{\rho_4}\sigma\right)(\xi,\rho_3)=&\ R^{\perp}(\xi,\rho_2)\left\{\triangledown^{\perp}_{\rho_4}\sigma(\xi,\rho_3)-\sigma\left(\triangledown_{\rho_4}\xi,\rho_3\right)-\sigma\left(\xi,\triangledown_{\rho_4}\rho_3\right)\right\}\\=&\ -R^{\perp}(\xi,\rho_2)\sigma\left(\triangledown_{\rho_4}\xi,\rho_3\right)\\=&\ R^{\perp}(\xi,\rho_2)\left\{\sigma(\phi h\rho_4,\rho_3)-\sigma(\rho_4,\rho_3)\right\},\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{C(\xi,\rho_2)\rho_4}\sigma\right)(\xi,\rho_3) = & \bigtriangledown^{\perp}_{C(\xi,\rho_2)\rho_4}\sigma(\xi,\rho_3)-\sigma\left(\bigtriangledown_{C(\xi,\rho_2)\rho_4}\xi,\rho_3\right)-\sigma\left(\xi,\bigtriangledown_{C(\xi,\rho_2)\rho_4}\rho_3\right)\\
= & \sigma\left(\phi^2C(\xi,\rho_2)\rho_4+\phi hC(\xi,\rho_2)\rho_4,\rho_3\right)\\
= & \eta(\rho_4)\left\{\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_2,\rho_3)+\mu\sigma(h\rho_2,\rho_3)\right.\\
& +\nu\sigma(\phi h\rho_2,\rho_3)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\phi\sigma(h\rho_2,\rho_3)\\
& \left.+(\kappa+1)\mu\phi\sigma(\rho_2,\rho_3)-(\kappa+1)\nu\sigma(\rho_2,\rho_3)\right\}
\end{aligned} \tag{3.24}$$

$$\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(C(\xi,\rho_2)\xi,\rho_3)=\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)\left(\left[\kappa-\frac{r}{2n(2n+1)}\right][\eta(\rho_2)\xi-\rho_2]-\mu h\rho_2-\nu\phi h\rho_2,\rho_3\right) \tag{3.25}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\xi,C(\xi,\rho_2)\rho_3) = & \nabla^{\perp}_{\rho_4}\sigma(\xi,C(\xi,\rho_2)\rho_3)-\sigma\left(\nabla_{\rho_4}\xi,C(\xi,\rho_2)\rho_3\right)\\
& -\sigma\left(\xi,\nabla_{\rho_4}C(\xi,\rho_2)\rho_3\right)\\
= & -\sigma\left(\nabla_{\rho_4}\xi,C(\xi,\rho_2)\rho_3\right)\\
= & \sigma\left(\phi^2\rho_4+\phi h\rho_4,C(\xi,\rho_2)\rho_3\right)\\
= & \eta(\rho_3)\left\{\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_4,\rho_2)+\mu\sigma(\rho_4,h\rho_2)\right.\\
& +\nu\phi\sigma(\rho_4,h\rho_2)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\phi h\rho_4,\rho_2)\\
& \left.+\mu(\kappa+1)\phi\sigma(\rho_4,\rho_2)-\nu(\kappa+1)\sigma(\rho_4,\rho_2)\right\},
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{(\xi\wedge_g\rho_2)\rho_4}\sigma\right)(\xi,\rho_3) = & \bigtriangledown^{\perp}_{(\xi\wedge_g\rho_2)\rho_4}\sigma(\xi,\rho_3)-\sigma\left(\bigtriangledown_{(\xi\wedge_g\rho_2)\rho_4}\xi,\rho_3\right)-\sigma\left(\xi,\bigtriangledown_{(\xi\wedge_g\rho_2)\rho_4}\rho_3\right)\\
= & \sigma\left(\phi^2(\xi\wedge_g\rho_2)\rho_4+\phi h(\xi\wedge_g\rho_2)\rho_4,\rho_3\right)\\
= & \eta(\rho_4)\{\sigma(\rho_2,\rho_3)-\phi\sigma(h\rho_2,\rho_3)\},
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)((\xi\wedge_g\rho_2)\xi,\rho_3) = & \left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\eta(\rho_2)\xi-\rho_2,\rho_3)\\
= & \left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\eta(\rho_2)\xi,\rho_3)-\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\rho_2,\rho_3)\\
= & -\sigma\left(\nabla_{\rho_4}\eta(\rho_2)\xi,\rho_3\right)-\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\rho_2,\rho_3)\\
= & -\sigma\left(\rho_4[\eta(\rho_2)]\xi+\eta(\rho_2)\nabla_{\rho_4}\xi,\rho_3\right)-\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\rho_2,\rho_3)\\
= & \eta(\rho_2)\{\sigma(\phi h\rho_4,\rho_3)-\sigma(\rho_4,\rho_3)\}-\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\rho_2,\rho_3)
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\left(\tilde{\bigtriangledown}_{\rho_4}\sigma\right)(\xi,(\xi\wedge_g\rho_2)\rho_3) = & -\sigma\left(\nabla_{\rho_4}\xi,(\xi\wedge_g\rho_2)\rho_3\right)\\
= & \sigma\left(\phi^2\rho_4+\phi h\rho_4,g(\rho_2,\rho_3)\xi-g(\xi,\rho_3)\rho_2\right)\\
= & \eta(\rho_3)\{\sigma(\rho_4,\rho_2)-\sigma(\phi h\rho_4,\rho_2)\}
\end{aligned} \tag{3.29}$$

If we substitute (3.22), (3.23), (3.24), (3.25), (3.26), (3.27), (3.28) in (3.21), we obtain

$$\begin{aligned}
&R^{\perp}(\xi,\rho_2)\{\sigma(\phi h\rho_4,\rho_3)-\sigma(\rho_4,\rho_3)\}-\eta(\rho_4)\Big\{\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_2,\rho_3)+\mu\sigma(h\rho_2,\rho_3)\\
&+\nu\sigma(\phi h\rho_2,\rho_3)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\phi\sigma(h\rho_2,\rho_3)+(\kappa+1)\mu\phi\sigma(\rho_2,\rho_3)-(\kappa+1)\nu\sigma(\rho_2,\rho_3)\}\\
&-(\tilde{\nabla}_{\rho_4}\sigma)\left(\left[\kappa-\tfrac{r}{2n(2n+1)}\right][\eta(\rho_2)\xi-\rho_2]-\mu h\rho_2-\nu\phi h\rho_2,\rho_3\right)\\
&-\eta(\rho_3)\left\{\begin{array}{l}\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_4,\rho_2)+\mu\sigma(\rho_4,h\rho_2)+\nu\phi\sigma(\rho_4,h\rho_2)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\phi h\rho_4,\rho_2)\\
+\mu(\kappa+1)\phi\sigma(\rho_4,\rho_2)-\nu(\kappa+1)\sigma(\rho_4,\rho_2)\end{array}\right\}\\
&=-F_3\{\eta(\rho_4)\{\sigma(\rho_2,\rho_3)-\phi\sigma(h\rho_2,\rho_3)\}+\eta(\rho_2)\{\sigma(\phi h\rho_4,\rho_3)-\sigma(\rho_4,\rho_3)\}-(\tilde{\nabla}_{\rho_4}\sigma)(\rho_2,\rho_3)\\
&\quad+\eta(\rho_3)\{\sigma(\rho_4,\rho_2)-\sigma(\phi h\rho_4,\rho_2)\}\}
\end{aligned}\tag{3.30}$$

If we choose $\rho_3=\xi$ in (3.30), we get

$$\begin{aligned}
&-(\tilde{\nabla}_{\rho_4}\sigma)\left(\left[\kappa-\tfrac{r}{2n(2n+1)}\right][\eta(\rho_2)\xi-\rho_2]-\mu h\rho_2-\nu\phi h\rho_2,\xi\right)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_4,\rho_2)-\mu\sigma(\rho_4,h\rho_2)\\
&-\nu\phi\sigma(\rho_4,h\rho_2)+\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\phi h\rho_4,\rho_2)-\mu(\kappa+1)\phi\sigma(\rho_4,\rho_2)+\nu(\kappa+1)\sigma(\rho_4,\rho_2)\\
&=-F_3\{-(\tilde{\nabla}_{\rho_4}\sigma)(\rho_2,\xi)+\sigma(\rho_4,\rho_2)-\sigma(\phi h\rho_4,\rho_2)\}.
\end{aligned}\tag{3.31}$$

By direct calculations, one can easily see that

$$\begin{aligned}
&(\tilde{\nabla}_{\rho_4}\sigma)\left(\left[\kappa-\tfrac{r}{2n(2n+1)}\right][\eta(\rho_2)\xi-\rho_2]-\mu h\rho_2-\nu\phi h\rho_2,\xi\right)\\
&=\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\sigma(\rho_2,\rho_4)+\mu\sigma(h\rho_2,\rho_4)+\nu\phi\sigma(\rho_4,h\rho_2)-\left[\kappa-\tfrac{r}{2n(2n+1)}\right]\phi\sigma(\rho_2,h\rho_4)\\
&\quad+\mu(\kappa+1)\phi\sigma(\rho_4,\rho_2)-\nu(\kappa+1)\sigma(\rho_4,\rho_2),
\end{aligned}\tag{3.32}$$

and

$$(\tilde{\nabla}_{\rho_4}\sigma)(\rho_2,\xi)=\phi\sigma(h\rho_4,\rho_2)-\sigma(\rho_4,\rho_2).\tag{3.33}$$

If (3.32) and (3.33) are out in (3.31), we obtain

$$\left[F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)+(\nu-\mu\phi)(\kappa+1)\right]\sigma(\rho_4,\rho_2)-\left[\left(F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)\right)\phi+(\mu+\phi\nu)\right]\sigma(\rho_4,h\rho_2)=0.\tag{3.34}$$

Substituting $h\rho_2$ instead of ρ_2 in (3.34), we can easily see that

$$\begin{aligned}
&\left[F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)+(\nu-\mu\phi)(\kappa+1)\right]\sigma(\rho_4,h\rho_2)\\
&-\left[\left(F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)\right)\phi+(\mu+\phi\nu)\right](\kappa+1)\sigma(\rho_4,\rho_2)=0.
\end{aligned}\tag{3.35}$$

From common solutions of (3.34) and (3.35), we can infer

$$\begin{aligned}
&\Big\{\left[F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)+(\nu-\mu\phi)(\kappa+1)\right]^2\\
&+\left[\left(F_3-\left(\kappa-\tfrac{r}{2n(2n+1)}\right)\right)\phi+(\mu+\phi\nu)\right]^2(\kappa+1)\Big\}\sigma(\rho_4,\rho_2)=0
\end{aligned}\tag{3.36}$$

This implies that N is either totally geodesic or

$$F_3=\left[\kappa-\frac{r}{2n(2n+1)}\right]\mp\sqrt{(\kappa+1)(\nu^2-\mu^2)},\mu\cdot\nu(\kappa+1)=0$$

This completes of the proof. □

Corollary 3.8. *Let N be an invariant pseudoparallel submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. Then N is concircular $2-$semiparallel if and only if N is totally geodesic provided*

$$\left[\kappa-\frac{r}{2n(2n+1)}\right]^{2}-(\kappa+1)\left(\nu^{2}-\mu^{2}\right)\neq 0 \; or \; (\kappa+1)\mu\nu\neq 0.$$

Equivalent to the definition of concircular $2-$Ricci generalized pseudoparallel given above, it can be said that there is a function F_4 on the set
$M_4=\left\{x\in N|\,S(x)\neq\tilde{\nabla}\sigma(x)\right\}$ such that

$$C\cdot\tilde{\nabla}\sigma=F_4 Q\left(S,\tilde{\nabla}\sigma\right).$$

If $F_4=0$ specifically, N is called a concircular $2-$Ricci generalized semiparallel submanifold.

Theorem 3.9. *Let N be the invariant submanifold of the $(2n+1)-$dimensional an almost Kenmotsu $(\kappa,\mu,\nu)-$space $\tilde{N}^{2n+1}(\phi,\xi,\eta,g)$. If N is concircular $2-$Ricci generalized pseudoparallel submanifold, then N is either a total geodesic submanifold or*

$$F_4=\frac{1}{2n}\left(1\mp\frac{2n(2n+1)}{2n(2n+1)\kappa-r}\sqrt{(\kappa+1)(\nu^{2}-\mu^{2})}\right),\mu\cdot\nu(\kappa+1)=0.$$

Proof. The proof of the theorem can be easily done similar to the proof of the previous theorem. □

4. Conclusion

In this article, pseudoparallel submanifolds for almost Kenmotsu $(\kappa,\mu,\nu)-$space are investigated. The almost Kenmotsu $(\kappa,\mu,\nu)-$space is considered on the concircular curvature tensor. Submanifolds of these manifolds with properties such as concircular pseudoparallel, concircular $2-$pseudoparallel, concircular Ricci generalized pseudoparallel, and concircular $2-$Ricci generalized pseudoparallel has been characterized. Necessary and sufficient conditions are given for the invariant submanifolds of almost Kenmotsu $(\kappa,\mu,\nu)-$space to be total geodesic according to the behavior of the κ,μ,ν functions.

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